

LYAPUNOV GRAPHS OF NONSINGULAR SMALE FLOWS ON $S^1 \times S^2$

BIN YU

ABSTRACT. In this paper, following J. Franks' work on Lyapunov graphs of nonsingular Smale flows on S^3 , we study Lyapunov graphs of nonsingular Smale flows on $S^1 \times S^2$. More precisely, we determine necessary and sufficient conditions on an abstract Lyapunov graph to be associated with a nonsingular Smale flow on $S^1 \times S^2$. We also study the singular type vertices in Lyapunov graphs of nonsingular Smale flows on 3-manifolds.

1. INTRODUCTION

Lyapunov graphs were first introduced in dynamics by J. Franks in his paper [7]. For a smooth flow $\phi_t : M \rightarrow M$ with a Lyapunov function $f : M \rightarrow R$, a Lyapunov graph L is a rather natural object. The idea is to construct an oriented graph by identifying to a point each component of $f^{-1}(c)$ for each $c \in R$. A Lyapunov graph gives a global picture of how the basic sets of a smooth flow are situated on the underlying manifold. It is important to note that a Lyapunov graph does not always determine a unique flow up to topological equivalence.

In [7], J. Franks used Lyapunov graph to classify nonsingular Smale flows (abbreviated as *NS flows*) on S^3 . More precisely, J. Franks determined necessary and sufficient conditions on an abstract Lyapunov graph to be associated with an NS flow on S^3 . Following the idea of J. Franks, K. de Rezende and her cooperators classified Smale flows on S^3 ([11]), gradient-like flows on 3-manifolds ([12]) and smooth flows on surfaces ([13]).

NS flows were first introduced by J. Franks in the 1980s, see [5], [6], and [7]. An NS flow is a structurally stable flow with one-dimensional invariant sets and without singularities. More restricted Smale flows, namely nonsingular Morse-Smale flows (abbreviated as *NMS flows*), have been effectively studied by associating NMS flows with a kind of combinatorial tool, i.e. round handle decomposition, see [1], [9] and [15]. However, for general NS flows, the situation becomes more complicated. M. Sullivan [14] and the author [16] used template and knot theory to describe more embedding information of a special type of nonsingular Smale flows on S^3 . J. Franks ([5], [6]) used homology to describe some embedding information of NS flows on 3-manifolds.

More recently, F. Béguin and C. Bonatti [2] gave several new concepts and results to describe the behavior of a neighborhood of a basic set of a Smale flow on a 3-manifold. Their paper is an important step in the classification, up to

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topological equivalence, of NS flows on 3-manifolds. It provided useful canonical neighborhoods of one-dimensional hyperbolic basic sets on 3-manifolds, i.e. filtrating neighborhoods. We note that the same concept was also discovered by J. Franks in [7], who named it “building block”.

The work of F. Béguin and C. Bonatti [2] indicates that the topological structures of all Lyapunov graphs of a Smale flow on a 3-manifold are the same and a filtrating neighborhood is always associated with a vertex in a Lyapunov graph (More details could be found in Section 6). These facts convinces us that a generalization of J. Franks’ work for all 3-manifolds is an efficient way of understanding NS flows on 3-manifolds. In this direction, N. Oka has made some progress in [10]. He generalized J. Franks’ work to all lens spaces with odd order fundamental group. The results and the techniques in this paper are similar to J. Franks’. For example, J. Franks proved that a Lyapunov graph of any NS flow on S^3 must be a tree and the weight of any edge of L is 1, i.e., the regular level sets of any NS flow on S^3 must be tori. The same phenomena occur in the study of NS flows on a lens space with odd order fundamental group.

In this paper, we mainly study Lyapunov graphs of NS flows on $S^1 \times S^2$. The reasons why we choose $S^1 \times S^2$ are:

- a Lyapunov graph of an NS flow on $S^1 \times S^2$ may not be a tree;
- the weight of an edge of a Lyapunov graph of an NS flow may not be 1;
- for our purpose, the topology of $S^1 \times S^2$ is well understood.

We use three theorems, i.e., Theorem 5.1, Theorem 5.2 and Theorem 5.5 to determine necessary and sufficient conditions on an abstract Lyapunov graph to be associated with an NS flow on $S^1 \times S^2$. The fact that we need three theorems is due to some topological information of $S^1 \times S^2$ (see Proposition 4.4).

In addition, to generalize J. Franks’ work for all 3-manifolds, we find that it is important to consider a kind of vertices, i.e., singular vertices (the definition can be found in Section 6). In the end of this paper, we study singular vertices of NS flows on irreducible 3-manifolds. The main result (Proposition 6.3) is that the number of the singular vertices in a Lyapunov graph associated with an NS flow on an irreducible, closed orientable 3-manifold is restricted by the Haken number of the 3-manifold.

This paper is organized as follows. In Section 2, we give some definitions and detailed background knowledge for Lyapunov graphs of NS flows on 3-manifolds. In Section 3, we discuss some connections between Smale flows and homology. In Section 4, we prove some general properties about NS flows on $S^1 \times S^2$. In Section 5, we state and prove the main theorems of this paper. In Section 6 we obtain some results on singular vertices of Lyapunov graphs of NS flows on 3-manifolds.

2. LYAPUNOV GRAPH OF NONSINGULAR SMALE FLOW

A smooth flow ϕ_t on a compact manifold M is called a *Smale flow* if:

- (1) the chain recurrent set $R(\phi_t)$ has hyperbolic structure;
- (2) $\dim(R(\phi_t)) \leq 1$;
- (3) ϕ_t satisfies the transversality condition and noncyclic condition.

If a Smale flow ϕ_t has no singularities, we call ϕ_t a *nonsingular Smale flow* (abbreviated as an *NS flow*). In particular, if $R(\phi_t)$ consists entirely of closed orbits, ϕ_t is called a *nonsingular Morse-Smale flow* (abbreviated as an *NMS flow*). For a

Smale flow ϕ_t , if $R(\phi_t)$ consists entirely of singularities, ϕ_t is called a *gradient-like flow*.

A diffeomorphism f on a closed manifold M is called a *Morse-Smale* diffeomorphism if:

- (1) the chain recurrent set $R(f)$ has hyperbolic structure and consists entirely of singularities;
- (2) for any $x, y \in R(f)$, the stable manifold $W^s(x)$ is transverse to the unstable manifold $W^u(y)$.

A Morse-Smale diffeomorphism f is called a *gradient-like* diffeomorphism if for any $x, y \in R(f)$, $W^s(x) \cap W^u(y) \neq \emptyset$ forces that $\dim W^s(x) > \dim W^s(y)$.

Definition 2.1. An *abstract Lyapunov graph* is a finite, connected and oriented graph L satisfying the following two conditions:

- (1) L possesses no oriented cycles;
- (2) each vertex of L is labeled with a chain recurrent flow on a compact space.

Theorem 2.2. (*R. Bowen [3]*)

If ϕ_t is a flow with hyperbolic chain recurrent set and Λ is a 1-dimensional basic set, then ϕ_t restricted to Λ is topologically equivalent to the suspension of a basic subshift of finite type (i.e., a subshift associated with an irreducible matrix (abbreviated as SSFT)).

This theorem enables us to label a vertex of the (abstract) Lyapunov graph that represents the 1-dimensional basic sets of a Smale flow with the suspension of an SSFT $\sigma(A)$. For simplicity we will label the vertex with the nonnegative integer irreducible matrix A . For a vertex labeled with matrix $A = (a_{ij})$, let $B = (b_{ij})$, where $b_{ij} \in \mathbb{Z}/2$ and $b_{ij} \equiv a_{ij} \pmod{2}$ and $k = \dim \ker((I - B) : F_2^m \rightarrow F_2^m)$, $F_2 = \mathbb{Z}/2$. The number of incoming (outgoing) edges is denoted by e^+ (e^-). Denote the weight of an edge of a Lyapunov graph by the genus of the regular level set of the edge. Let g_j^+ (g_j^-) be the weight on an incoming (outgoing) edge of the vertex. In this paper, we always denote the initial vertex and the terminal vertex of an oriented edge E by $i(E)$ and $t(E)$ respectively. Whenever we talk about a Lyapunov graph L of an NS flow on a 3-manifold M , L is always associated with a Lyapunov function $g : M \rightarrow R$ and a map $h : M \rightarrow L$ such that $g = \pi \circ h$. Here $\pi : L \rightarrow R$ is the natural projection.

The following classification theorem is due to J. Franks [7]:

Theorem 2.3. Let L be an abstract Lyapunov graph. L is associated with an NS flow ϕ_t on S^3 if and only if the following conditions hold.

- (1) The underlying graph L is a tree with exactly one edge attached to each closed orbit sink or closed orbit source vertex. The weight of any edge of L is 1.
- (2) If a vertex is labeled with an SSFT with matrix $A_{m \times m}$, then

$$0 < e^+ \leq k + 1, 0 < e^- \leq k + 1 \text{ and} \\ k + 1 \leq e^+ + e^-.$$

K. Rezende [11] generalized J. Franks' theorem to Smale flows on S^3 :

Theorem 2.4. Let L be an abstract Lyapunov graph. L is associated with a Smale flow ϕ_t on S^3 if and only if the following conditions hold.

(1) The underlying graph L is a tree with exactly one edge attached to each sink or source vertex.

(2) If a vertex is labeled with an SSFT with matrix $A_{m \times m}$, then we have

$$\begin{aligned} e^+ &> 0, e^- > 0, \\ k+1 - \sum_{i=1}^{e^-} g_i^- &\leq e^+ \leq k+1 \text{ and} \\ k+1 - \sum_{j=1}^{e^+} g_j^+ &\leq e^- \leq k+1. \end{aligned}$$

(3) All vertices must satisfy the Poincaré-Hopf condition. Namely, if a vertex is labeled with a singularity of index r , then

$$(-1)^r = e^+ - e^- - \sum g_j^+ + \sum g_i^-,$$

and if a vertex is labeled with a suspension of an SSFT or a periodic orbit, then

$$0 = e^+ - e^- - \sum g_j^+ + \sum g_i^-.$$

The following theorem is due to [4]:

Theorem 2.5. Suppose a closed orientable 3-manifold M admits an NS flow ϕ_t with Lyapunov graph L , then $\beta_1(L) \leq \beta_1(M)$. Here $\beta_1(L)$ and $\beta_1(M)$ are the first Betti numbers of L and M respectively.

Throughout this paper, if V is a manifold, mV means the connected sum of m copies of V . The following theorem ([17]) tells us some relation between the topology of 3-manifolds and the weights of edges of Lyapunov graphs of NS flows:

Theorem 2.6. Suppose a closed orientable 3-manifold M admits an NS flow ϕ_t with Lyapunov graph L . Then M admits an NS flow which has a regular level set homeomorphic to $(n+1)T^2$ ($n \in \mathbb{Z}, n \geq 0$) if and only if $M = M' \# nS^1 \times S^2$. Here M' is any closed orientable 3-manifold. Furthermore, $\beta_1(L) \geq n$.

3. HOMOLOGY AND SMALE FLOWS

Definition 3.1. If $\phi_t : M \rightarrow M$ is a flow with hyperbolic chain recurrent set and its basic sets are $\{\Lambda_i\}$ ($i = 1, \dots, n$), then a *filtration* associated with ϕ_t is a collection of submanifolds $M_0 \subset M_1 \subset \dots \subset M_n = M$ such that

- (1) $\phi_t(M_i) \subset \text{Int } M_i$, for any $t > 0$;
- (2) $\Lambda_i = \cap_{t=-\infty}^{\infty} \phi_t(M_i - M_{i-1})$.

The following theorem is due to Bowen and J. Franks [6]:

Theorem 3.2. Let ϕ_t be a Smale flow and $M_i, i = 1, \dots, n$ be a filtration associated with ϕ_t . Suppose $\Lambda_i = \cap_{t=-\infty}^{\infty} \phi_t(M_i - M_{i-1})$ is an index u basic set labeled with a matrix $A_{n \times n}$, then:

$$\begin{aligned} H_k(M_i, M_{i-1}, F_2) &\cong 0, \text{ if } k \neq u, u+1; \\ H_u(M_i, M_{i-1}, F_2) &\cong F_2^n / (I - B)F_2^n; \\ H_{u+1}(M_i, M_{i-1}, F_2) &\cong \ker((I - B) \text{ on } F_2^n). \end{aligned}$$

Let M be a closed orientable 3-manifold, ϕ_t be a Smale flow on M and g be a Lyapunov function associated with ϕ_t . Assume that $c \in \mathbb{R}$ is a singular value and $g^{-1}(c)$ is associated to a basic set labeled with a matrix $B_{n \times n}$. Set $X = g^{-1}((-\infty, c + \epsilon])$, $Y = g^{-1}([c + \epsilon, +\infty))$ and $Z = g^{-1}((-\infty, c - \epsilon])$. Throughout this paper $k = \dim \ker(I - B)$. The homology coefficients in this paper shall be taken in $F_2 = \mathbb{Z}/2$.

Proposition 3.3. (1) $e^+ \leq k + 1 + \dim H_2(Z) + \dim H_2(Y)$;
 (2) $e^- \leq k + 1 + \dim H_2(Z) + \dim H_2(Y)$;
 (3) $k \leq e^+ - 1 + \dim H_1(M) + \dim H_1(Z) - \dim H_2(Z)$;
 (4) $k \leq e^- - 1 + \dim H_1(M) + \dim H_1(Y) - \dim H_2(Y)$.

Proof. We consider the Mayer-Vietoris exact sequence:

$$(3.1) \quad \begin{aligned} H_3(X) \oplus H_3(Y) &\rightarrow H_3(X \cup Y) \rightarrow H_2(X \cap Y) \xrightarrow{d_*} H_2(X) \oplus H_2(Y) \\ &\xrightarrow{c_*} H_2(X \cup Y) \end{aligned}$$

and the exact sequence for (X, Z) :

$$(3.2) \quad H_3(X, Z) \rightarrow H_2(Z) \rightarrow H_2(X) \xrightarrow{b_*} H_2(X, Z) \rightarrow H_1(Z) \rightarrow H_1(X).$$

Both X and Y are compact three manifolds with boundary, so $H_3(X) \cong H_3(Y) \cong 0$. $X \cup Y = M$, so $H_3(X \cup Y) \cong F_2$. $X \cap Y$ is composed of e^+ closed orientable surfaces, so $H_2(X \cap Y) \cong e^+ F_2$. Since (3.1) is an exact sequence, $\dim \operatorname{Im} d_* = \dim \ker c_* \leq \dim H_2(X) + \dim H_2(Y)$. It implies that $\dim \operatorname{Im} d_* = e^+ - 1$. Therefore, we have $e^+ - 1 \leq \dim H_2(X) + \dim H_2(Y)$. By the exact sequence (3.2), we have $\dim H_2(X) \leq \dim H_2(Z) + \dim H_2(X, Z)$. By Theorem 3.2, we have $\dim H_2(X, Z) = \dim \ker(I - B) = k$. Therefore, (1) in Proposition 3.3 is proved. (2) in Proposition 3.3 can be proved similarly.

By Theorem 3.2, we have $H_3(X, Z) = 0$. Therefore, by the exact sequence (3.2),

$$(3.3) \quad k = \dim H_2(X, Z) \leq \dim H_2(X) + \dim H_1(Z) - \dim H_2(Z).$$

Now we consider the reduced exact sequence for (M, Y) :

$$(3.4) \quad \widetilde{H}_1(Y) \rightarrow \widetilde{H}_1(M) \rightarrow \widetilde{H}_1(M, Y) \rightarrow \widetilde{H}_0(Y) \rightarrow \widetilde{H}_0(M).$$

Since $\widetilde{H}_0(M) = 0$ and $\dim \widetilde{H}_0(Y) = e^+ - 1$, by the exact sequence (3.4), we have $\dim H_1(M, Y) \leq e^+ - 1 + \dim H_1(M)$. On the other hand, by Lefschetz duality, $H_1(M, Y) \cong H_2(M - Y, M - M) = H_2(X)$. Therefore, $\dim H_2(X) \leq e^+ - 1 + \dim H_1(M)$. By (3.3), $k \leq e^+ - 1 + \dim H_1(M) + \dim H_1(Z) - \dim H_2(Z)$. Therefore, (3) in Proposition 3.3 is proved. (4) in Proposition 3.3 can be proved similarly. \square

Lemma 3.4. Let K be a knot in S^3 and K^c be the complement space of K in S^3 , then $H_1(K^c) \cong F_2$ and $H_2(K^c) \cong 0$.

Proof. By Lefschetz duality theorem, we have $H_1(K^c) \cong H_2(S^3, K)$ and $H_2(K^c) \cong H_1(S^3, K)$. By the exact sequence: $H_2(S^3) \rightarrow H_2(S^3, K) \rightarrow H_1(K) \rightarrow H_1(S^3) \rightarrow H_1(S^3, K)$, it is easy to show that $H_2(S^3, K) \cong F_2$ and $H_1(S^3, K) \cong 0$. Therefore, $H_1(K^c) \cong F_2$ and $H_2(K^c) \cong 0$. \square

Corollary 3.5. If each component of Y and Z is homeomorphic to a knot complement, then $e^+ \leq k + 1$, $e^- \leq k + 1$ and $k + 1 \leq e^+ + e^- + \dim H_1(M)$.

Proof. It follows easily from Proposition 3.3 and Lemma 3.4. \square

4. SOME PROPERTIES ON LYAPUNOV GRAPHS OF NS FLOWS ON $S^1 \times S^2$

Proposition 4.1. Let $i : T^2 \hookrightarrow S^1 \times S^2$ be an embedding map and $i_* : \pi_1(T^2) \rightarrow \pi_1(S^1 \times S^2)$ be the homomorphism induced by i . One and only one of the following three possibilities occurs.

- (1) If $i(T^2)$ is inseparable in $S^1 \times S^2$, then $S^1 \times S^2 - i(T^2) \cong (O \sqcup K)^c$. Here $(O \sqcup K)^c$ is the compliment of a two-component-link in S^3 , where O is a trivial knot and K can be any knot unlinked with O .
- (2) If $i(T^2)$ is separable in $S^1 \times S^2$ and $\text{Im } i_* = 0$, then

$$S^1 \times S^2 - i(T^2) \cong K^c \sqcup (S^1 \times D^2) \# (S^1 \times D^2)$$

or

$$S^1 \times S^2 - i(T^2) \cong (S^1 \times D^2) \sqcup (S^1 \times D^2) \# K^c.$$

Here $S^1 \times D^2$ is an open solid torus and K can be any knot.

- (3) If $i(T^2)$ is separable in $S^1 \times S^2$ and $\text{Im } i_* \neq 0$, then $i(T^2)$ bounds a solid torus in $S^1 \times S^2$.

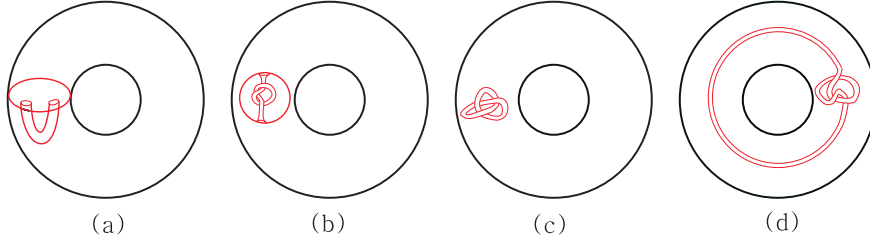


FIGURE 1.

Proof. Since $\pi_1(T^2) \cong 2\mathbb{Z}$ and $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$, $\ker i_* \neq 0$. Therefore, by Dehn's lemma (see [He]), there exists a nontrivial simple closed curve $c \subset i(T^2)$ such that c bounds a disk D in $S^1 \times S^2 - i(T^2)$. Cutting $i(T^2)$ along a ribbon neighborhood $N(c)$ of c and pasting two copies of D to $\partial(i(T^2) - N(c))$, we obtain a sphere S_0^2 . One and only one of the following three possibilities occurs.

- (1) If $i(T^2)$ is inseparable in $S^1 \times S^2$, then S_0^2 is also inseparable in $S^1 \times S^2$. Therefore, S_0^2 is isotopic to $\{pt\} \times S^2$ where pt is a point in S^1 . So it is easy to show that $S^1 \times S^2 - i(T^2) \cong (O \sqcup K)^c$. This case is illustrated by Figure 1 (a).
- (2) If $i(T^2)$ is separable in $S^1 \times S^2$ and $\text{Im } i_* = 0$, then S_0^2 bounds a 3-ball in $S^1 \times S^2$. Moreover,
 - (a) if c is in the 3-ball, then $S^1 \times S^2 - i(T^2) \cong K^c \sqcup (S^1 \times D^2) \# (S^1 \times D^2)$ (see Figure 1 (b));
 - (b) if c is not in the 3-ball, then $S^1 \times S^2 - i(T^2) \cong (S^1 \times D^2) \sqcup (S^1 \times D^2) \# K^c$ (see Figure 1 (c)).
- (3) If $i(T^2)$ is separable in $S^1 \times S^2$ and $\text{Im } i_* \neq 0$, then S_0^2 bounds a 3-ball in $S^1 \times S^2$ and c is not in the 3-ball. Therefore, $i(T^2)$ bounds a solid torus in $S^1 \times S^2$, as shown in Figure 1 (d).

□

From the proposition above, it is easy to show the following corollary.

Corollary 4.2. Let $i : T^2 \hookrightarrow S^1 \times S^2$ be an embedding map such that $i(T^2)$ is separable in $S^1 \times S^2$, then $i(T^2)$ bounds a knot complement in $S^1 \times S^2 - i(T^2)$.

The following lemma is Theorem 4.7 in [11]. It is proved by using Poincaré-Hopf formula.

Lemma 4.3. Suppose ϕ_t is a smooth flow on an odd-dimensional manifold M which transverses outside to ∂M^- and transverses inside to ∂M^+ , where $\partial M = \partial M^+ \cup \partial M^-$. Then $\sum I_p = \frac{1}{2}(X(\partial M^+) - X(\partial M^-))$ where the summation is taken over all singularities in M , I_p is the index of the singularity p and X denotes the Euler characteristic.

Proposition 4.4. Let ϕ_t be an NS flow on $S^1 \times S^2$ with Lyapunov graph L .

- (1) If each regular level set of (ϕ_t, L) is separable, then each regular level set of (ϕ_t, L) is homeomorphic to a torus and L is a tree.
- (2) If at least one of the regular level sets of (ϕ_t, L) is inseparable, then there are exactly the following two possibilities.
 - (a) If Σ is homeomorphic to T^2 , then each regular level set of (ϕ_t, L) is homeomorphic to T^2 . Moreover, $\beta_1(L) = 1$.
 - (b) If Σ is not homeomorphic to T^2 , then Σ is homeomorphic to S^2 or $2T^2$ and $\beta_1(L) = 1$. Furthermore, there exist at least one regular level set which is homeomorphic to S^2 and at least one regular level set which is homeomorphic to $2T^2$.

Proof. Let Σ be a regular level set of (ϕ_t, L) .

Suppose each regular level set of (ϕ_t, L) is separable. For a regular level set Σ , $S^1 \times S^2 = M_1 \cup_{\Sigma} M_2$ where M_1 and M_2 are two compact 3-manifolds with boundaries. By Lemma 4.3, we have $X(\Sigma) = 0$. Therefore, Σ is homeomorphic to a torus. If L isn't a tree, then there exists a cycle in L . The inverse image of a regular point in the cycle is an inseparable regular level set in $S^1 \times S^2$. It contradicts our assumption. Therefore, (1) in Proposition 4.4 is proved.

Now we suppose there exists a regular level set Σ of (ϕ_t, L) which is inseparable.

If $\Sigma \cong T^2$, by Proposition 4.1, $N = S^1 \times S^2 - \Sigma \cong (O \cup K)^c$ where O is a trivial knot and K is an arbitrary knot unlinked with O . Attaching a standard solid torus neighborhood of a closed orbit attractor (round 0-handle) and a standard solid torus neighborhood of a closed orbit repeller (round 2-handle) to $(N, \phi_t|_N)$ suitably, we obtain an NS flow ψ_t on S^3 . By Theorem 2.3, each regular level set of ψ_t is homeomorphic to a torus. Therefore, each regular level set of ϕ_t is homeomorphic to T^2 . Since there exists an inseparable regular level set, it is obvious that $\beta_1(L) \geq 1$. On the other hand, by Theorem 2.5, $\beta_1(L) \leq 1$. Therefore, $\beta_1(L) = 1$. (a) of (2) in Proposition 4.4 is proved.

If Σ is not homeomorphic to a torus, by a similar argument, we can show that $\beta_1(L) = 1$. By Theorem 2.6, we have $\Sigma \cong S^2$ or $\Sigma \cong 2T^2$. Let $g : M \rightarrow R$ be a Lyapunov function associated with L and $x \in \mathbb{R}$ be a regular value. Suppose $h : M \rightarrow L$ is a map such that $g = \pi \circ h$. Here $\pi : L \rightarrow R$ is the natural projection. Suppose $g^{-1}(x) = \Sigma_1 \sqcup \Sigma_2 \sqcup \dots \sqcup \Sigma_s$. By Lemma 4.3, $\sum_{i=1}^s X(\Sigma_i) = 0$. Since for any $i \in \{1, \dots, s\}$, Σ_i is homeomorphic to one of S^2 , T^2 and $2T^2$, it is easy to show that there exist at least one regular level set which is homeomorphic to S^2 and at least one regular level set which is homeomorphic to $2T^2$. Hence (b) of (2) in Proposition 4.4 is proved. \square

Remark 4.5. The result in Proposition 4.4 (1) is still true if we generalize $S^1 \times S^2$ to any other closed 3-manifold M . The proof is similar. Indeed, if we change $S^1 \times S^2$ to any other closed 3-manifold $M = M' \# nS^1 \times S^2$ where M' is prime to $S^1 \times S^2$, by Theorem 2.6, each regular level set of (ϕ_t, L) is homeomorphic to one of $S^2, T^2, \dots, (n+1)T^2$.

Proposition 4.6. Let ϕ_t be an NS flow on $S^1 \times S^2$ with Lyapunov graph L such that each regular level set of ϕ_t is homeomorphic to a torus.

- (1) If there exists an inseparable regular level set, then for any basic set of ϕ_t , $e^+ \leq k+1$, $e^- \leq k+1$ and $k+1 \leq e^+ + e^-$.
- (2) If each regular level set of ϕ_t is separable, then for any basic set of ϕ_t , $e^+ \leq k+1$, $e^- \leq k+1$ and $k \leq e^+ + e^-$. Furthermore, there exists at most one basic set of ϕ_t which satisfies that $k = e^+ + e^-$.

Proof. If there exists an inseparable regular level set Σ in $S^1 \times S^2$, by Proposition 4.1, $N = S^1 \times S^2 - \Sigma \cong (O \cup K)^c$ where O is a trivial knot and K is a knot unlinked with O . Attaching a round 0-handle and a round 2-handle to $(N, \phi_t|_N)$ suitably, we obtain an NS flow ψ_t on S^3 . By Theorem 2.3, each basic set of ψ_t satisfies that $e^+ \leq k+1$, $e^- \leq k+1$ and $k+1 \leq e^+ + e^-$. Therefore, for any basic set of ϕ_t , $e^+ \leq k+1$, $e^- \leq k+1$ and $k+1 \leq e^+ + e^-$.

If each regular level set of ϕ_t is separable, by Proposition 4.4, each regular level set is homeomorphic to a torus and L is a tree. Let Σ be a regular level set of ϕ_t , by Corollary 4.2, Σ bounds a knot complement in $S^1 \times S^2 - \Sigma$. Let g be a Lyapunov function associated with L . Assume that $c \in \mathbb{R}$ is a singular value and $g^{-1}(c)$ is associated with a basic set Λ labeled with a matrix $B_{n \times n}$. Set $X = g^{-1}((-\infty, c+\epsilon])$, $Y = g^{-1}([c+\epsilon, +\infty))$ and $Z = g^{-1}((-\infty, c-\epsilon])$. If each component of Y and Z is a knot complement, by Corollary 3.5, $e^+ \leq k+1$, $e^- \leq k+1$ and $k \leq e^+ + e^-$. Otherwise, without loss of generality, we suppose that a component of Y , denoted by Y_1 , is not a knot complement. Therefore, $S^1 \times S^2 - Y_1$ is a knot complement. By an argument similar to the proof of (1) in Proposition 4.6, we have $e^+ \leq k+1$, $e^- \leq k+1$ and $k+1 \leq e^+ + e^-$. In summary, $e^+ \leq k+1$, $e^- \leq k+1$ and $k \leq e^+ + e^-$. We choose a regular level set for each edge of L and these regular level sets divide $S^1 \times S^2$ into m manifolds with boundary, N_1, \dots, N_m . Each N_i corresponds to a basic set Λ_i . Since each regular level set of ϕ_t bounds a knot complement, at most one of N_1, \dots, N_m is not a link complement. If there is one, we assume without loss of generality that it is N_1 . For an (N_j, Λ_j) ($j \neq 1$), by an argument similar to the proof of (1) in Proposition 4.6, $k+1 \leq e^+ + e^-$. Therefore, there exists at most one basic set of ϕ_t which satisfies that $k = e^+ + e^-$. \square

5. MAIN RESULTS: LYAPUNOV GRAPHS OF NS FLOWS ON $S^1 \times S^2$

In this section, we prove Theorem 5.1, Theorem 5.2 and Theorem 5.5, which give necessary and sufficient conditions on an abstract Lyapunov graph to be associated with an NS flow on $S^1 \times S^2$. The fact that we need three theorems is due to Proposition 4.4.

Theorem 5.1. *Let L be an abstract Lyapunov graph, then L is associated with an NS flow ϕ_t on $S^1 \times S^2$ such that there exists an inseparable regular level set which is homeomorphic to T^2 if and only if the following conditions hold.*

- (1) The underlying graph L is an oriented graph and $\beta_1(L) = 1$ with exactly one edge attached to each closed orbit sink or closed orbit source vertex. The weight of any edge of L is 1.
- (2) If a vertex is labeled with an SSFT with matrix $A_{m \times m}$, then

$$0 < e^+ \leq k+1, 0 < e^- \leq k+1 \text{ and} \\ k+1 \leq e^+ + e^-.$$

Proof. The necessity. Let ϕ_t be an NS flow on $S^1 \times S^2$ which satisfies that there exists an inseparable regular level set which is homeomorphic to T^2 . Suppose L is a Lyapunov graph of ϕ_t . Since there exists an inseparable regular level set which is homeomorphic to T^2 , by (a) of (2) in Proposition 4.4, we have $\beta_1(L) = 1$ and the weight of any edge of L is 1. By (1) in Proposition 4.6, we have $0 < e^+ \leq k+1$, $0 < e^- \leq k+1$ and $k+1 \leq e^+ + e^-$. The noncyclic condition of NS flow implies that L possesses no oriented cycles.

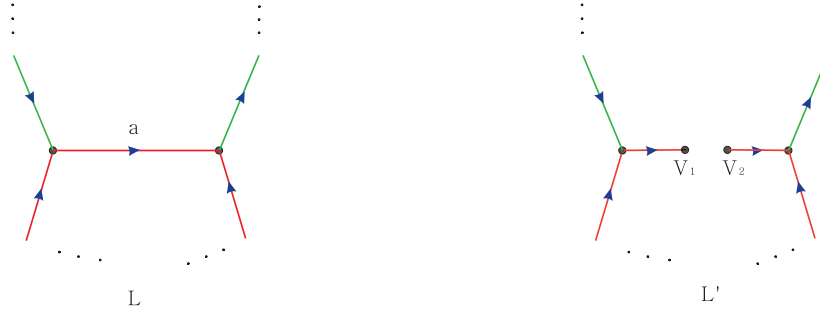


FIGURE 2.

The sufficiency. Let L be a Lyapunov graph which satisfies (1) and (2) of Theorem 5.1. Suppose a is an edge in L such that a belongs to the cycle of L . We can obtain a new Lyapunov graph L' by cutting L along a , then adding two vertices V_1 and V_2 which are labeled by a closed orbit sink and a closed orbit source respectively, as shown in Figure 2. It is easy to show that L' satisfies the sufficient condition of Theorem 2.3. Therefore, there exists an NS flow ψ_t on S^3 with Lyapunov graph L' . Suppose the sink labeled with V_1 and the source labeled with V_2 in ψ_t are K_1 and K_2 respectively. By the constructions of NS flows in [7], $K_1 \sqcup K_2$ is a two-component unlinked link such that K_1 and K_2 both are trivial knots. Cutting two standard solid torus neighborhoods $N(K_1)$ and $N(K_2)$ of K_1 and K_2 respectively, then pasting $S^3 - (N(K_1) \sqcup N(K_2))$ along $\partial N(K_1)$ and $\partial N(K_2)$ suitably, we obtain an NS flow ϕ_t on $S^1 \times S^2$ with Lyapunov graph L . \square

Theorem 5.2. *Let L be an abstract Lyapunov graph, then L is associated with an NS flow ϕ_t on $S^1 \times S^2$ such that there exists an inseparable regular level set which is homeomorphic to S^2 if and only if the following conditions hold.*

- (1) The underlying graph L is an oriented graph and $\beta_1(L) = 1$ with exactly one edge attached to each closed orbit sink or closed orbit source vertex. Suppose the cycle C of L is composed of n edges. Then C and L satisfy the following conditions.

- (a) The weight of an edge of C is 0 or 2. Moreover, there exist at least one weight 0 edge and one weight 2 edge in C .
 - (b) The orientation of a weight 0 edge in C is opposite to the orientation of a weight 2 edge in C .
 - (c) The weight of any other edge of L is 1.
- (2) If a vertex is labeled with an SSFT with matrix $A_{m \times m}$, then

$$0 < e^+ \leq k+1, 0 < e^- \leq k+1.$$

Moreover,

- (a) if there exist a weight 0 edge starting at the vertex and a weight 0 edge terminating at the vertex, then $k+2 \leq e^+ + e^-$;
- (b) if there exist a weight 2 edge starting at the vertex and a weight 2 edge terminating at the vertex, then $k \leq e^+ + e^-$;
- (c) otherwise, $k+1 \leq e^+ + e^-$.

Figure 3 shows two examples for $n = 6$.

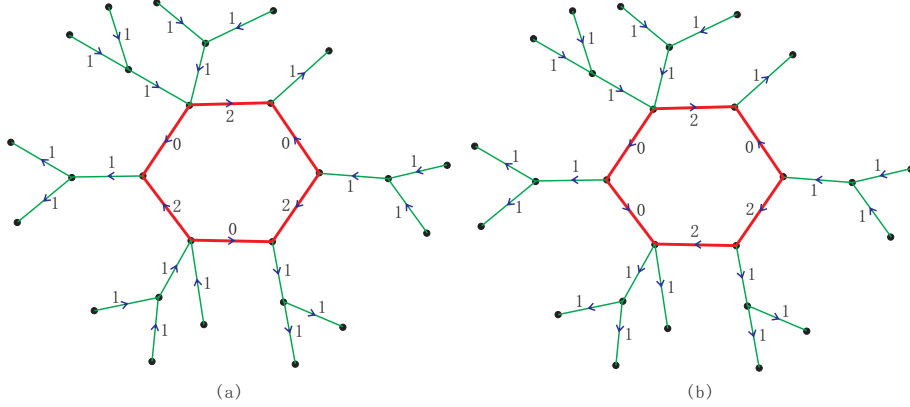


FIGURE 3.

Proof. The necessity. Let ϕ_t be an NS flow on $S^1 \times S^2$ which satisfies that there exists an inseparable regular level set which is homeomorphic to S^2 . Suppose L is a Lyapunov graph of ϕ_t . By (2) in Proposition 4.4, $\beta_1(L) = 1$. Therefore, there exists a unique cycle $C \subset L$. Let b be an edge in $\overline{L - C}$. By Lemma 4.3, it is easy to show that the weight of b is 1. Since there exists a inseparable regular level set which is homeomorphic to S^2 , by (b) in Proposition 4.4, the weight of an edge of C is 0 or 2 and there exist at least one weight 0 edge and one weight 2 edge in C . Let a and b be two adjacent edges in C . By Lemma 4.3 and Proposition 4.4, there are three cases:

- (1) a and b are both weight 0 edges. Moreover, the orientations of a and b in C are the same.
- (2) a and b are both weight 2 edges. Moreover, the orientations of a and b in C are the same.
- (3) The weight of a is 0 (2, resp.) and the weight of b is 2 (0, resp.). Moreover, the orientation of a in C is opposite to the orientation of b in C .

It follows easily from the above observations that the orientation of an weight 0 edge in C is opposite to the orientation of an weight 2 edge in C .

Let V be a vertex in L labeled with an SSFT with matrix $A_{m \times m}$. Suppose a is a weight 0 edge in L . We can obtain a new Lyapunov graph L' by cutting L along a , then pasting a singularity sink vertex V_1 and a singularity source vertex V_2 , as shown in Figure 2. Then L' is a Lyapunov graph of a Smale flow on S^3 . By Theorem 2.4, $0 < \max\{e^+, e^-\} \leq k + 1$, $k + 1 \leq \sum_{i=1}^{e^-} g_i^- + e^+$ and $k + 1 \leq \sum_{j=1}^{e^+} g_j^+ + e^-$. Therefore, by some easy computations, V satisfies (2) in Theorem 5.2.

The sufficiency. Let L be a Lyapunov graph which satisfies (1) and (2) of Theorem 5.2. Suppose a is a weight 0 edge in L . We can obtain a new Lyapunov graph L' by cutting L along a , then pasting a singularity sink vertex V_1 and a singularity source vertex V_2 , as shown in Figure 2. It is easy to show that L' satisfies the sufficient condition of Theorem 2.4. Therefore, there exists a Smale flow ψ_t on S^3 with Lyapunov graph L' . Suppose the singularity sink labeled with V_1 and the singularity source labeled with V_2 are s_1 and s_2 respectively. Cutting two standard 3-ball neighborhoods $N(s_1)$ and $N(s_2)$ of s_1 and s_2 respectively, then pasting $S^3 - (N(s_1) \sqcup N(s_2))$ along $\partial N(s_1)$ and $\partial N(s_2)$ suitably, we obtain an NS flow ϕ_t on $S^1 \times S^2$ with Lyapunov graph L . \square

The following lemma is due to J. Franks [7]:

Lemma 5.3. *Let $A_{n \times n}$ be a nonnegative irreducible integer matrix, then for any $N > 0$, there exists a nonnegative irreducible integer matrix $A' = (a'_{ij})_{m \times m}$ such that:*

- (1) *the SSFT with A is conjugate to the SSFT with A' ;*
- (2) *$a'_{ij} > N$ for any $i, j \in \{1, \dots, m\}$ and a'_{ij} is an even number if $i \neq j$.*

Lemma 5.4. *Let $A_{n \times n}$ be a nonnegative irreducible integer matrix. For any $e^+, e^- \in \mathbb{N}$ with $e^+ + e^- = k$, there exists an NS flow ψ_t on a compact 3-manifold N such that:*

- (1) *the chain recurrent set of ψ_t is conjugate to the SSFT with A ;*
- (2) *ψ_t is transverse to ∂N and moreover, the entrance set and the exit set of ψ_t on ∂N are composed of e^+ tori and e^- tori respectively;*
- (3) *N can be embedded into $S^1 \times S^2$ by an embedding map $i : N \rightarrow S^1 \times S^2$ such that $\overline{S^1 \times S^2 - i(N)} \cong (e^+ + e^-)(S^1 \times D^2)$.*

Proof. Step 1. Construct a gradient-like diffeomorphism f on S^2 .

We consider a 2-sphere S^2 with the spherical geometry. As Figure 4 shows, there exists a handle decomposition with two 0-handles, two 2-handles and two 1-handles. The handle decomposition corresponds to a gradient-like flow φ_t on S^2 with two index 0 singularities a_1 and a_2 , two index 2 singularities r_1 and r_2 and two index 1 singularities s_1 and s_2 . C_1 and C_2 are two orthogonal circles in S^2 . Figure 4 (a) and Figure 4 (b) both show the handle decomposition. In Figure 4 (b), S^2 is identified with $\mathbb{C} \cup \{\infty\}$ and $r_2 = \infty$. There exist two standard reflections T_1 and T_2 on S^2 leaving C_1 and C_2 fixed respectively. Set $f = T_2 \circ T_1 \circ \varphi_1 : S^2 \rightarrow S^2$. It is easy to check that f is a gradient-like diffeomorphism on S^2 with an index 0 orbit $\{a_1, a_2\}$ with periodic 2, an index 2 orbit $\{r_1, r_2\}$ with periodic 2 and two index 1 singularities $\{s_1\} \sqcup \{s_2\}$.

Step 2. Add handles and construct a Morse-Smale diffeomorphism ϕ_t on S^2 .

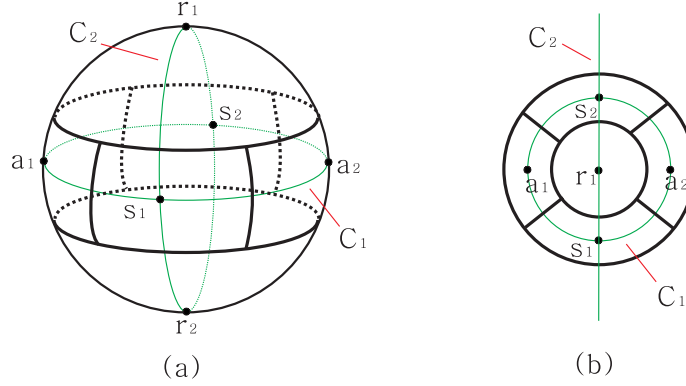


FIGURE 4.

Following J. Franks, we first introduce three types of handles, as shown in Figure 5. The first one is composed of a 0-handle and a 1-handle (a neighborhood of a sink and that of a saddle) and it is called an *SI handle*; the second one is composed of a 2-handle and a 1-handle and it is called an *SO handle* (a neighborhood of a source and a saddle); the last one doesn't admit any singularity and it is called an *NI handle* (it is called a nilpotent handle in [7]).

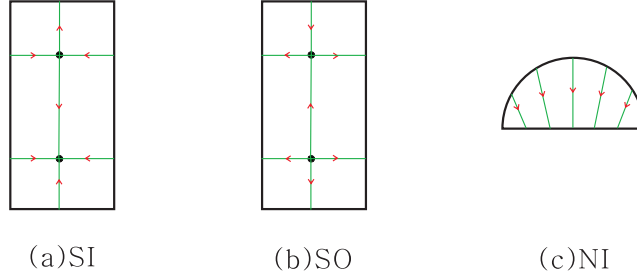


FIGURE 5.

Now we add handles to the handle decomposition in Step 1. In Figure 6, the disk with boundary L_3 is denoted by Y . We add $(e^+ - 1)$ SO handles, $(e^- - 1)$ SI handles and $(n - k)$ NI handles to the handle decomposition in Step 1 in the interior of Y . Suppose ϕ_t is a Morse-Smale diffeomorphism induced by the handle decomposition after adding handles. Figure 6 shows an example for $e^+ = e^- = 2$ and $n - k = 1$.

Step 3. Construct a Smale diffeomorphism g on S^2 .

Due to Lemma 5.3, in order to prove Lemma 5.4, we may assume that a_{ij} is an even number if $i \neq j$ for any $i, j \in \{1, \dots, n\}$. In Figure 6, we define $X = \overline{D_2 - D_1}$ where D_1 (D_2) is the disk with boundary L_1 (L_2). Moreover, set $X_0 = \overline{X - D}$ where D is a small standard neighborhood of the sinks and sources of the added handles. The number of the saddle singularities in X is $(e^+ - 1) + (e^- - 1) + 1 + 1 = e^+ + e^- = k$

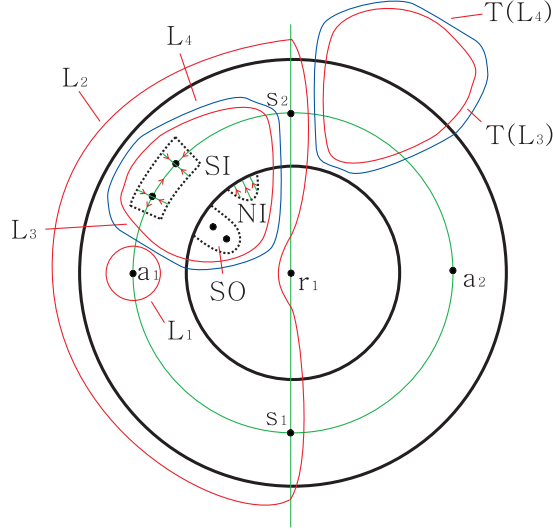


FIGURE 6.

and the number of NI handles is $n - k$. By the trick in the proof of Lemma 4.2 in [7], we obtain a Smale diffeomorphism g on S^2 such that:

- (1) if x is in $S^2 - X_0$ or in the right side of C_2 (see Figure 4), then $g(x) = \phi_1(x)$;
- (2) there exists only one basic set Λ in X_0 and Λ is a dimension 1 saddle basic set with matrix A ;
- (3) there exists a Markov partition of Λ with n rectangles $R_1 \cup \dots \cup R_n$ such that:
 - (a) each rectangle corresponds to a saddle singularity or an NI handle;
 - (b) the rectangles corresponding the saddle singularities in Y and the NI handles are in the interior of Y ;
 - (c) the geometric intersection matrix of the rectangles is A .

Step 4. Construct an NS flow ψ_t on a compact 3-manifold N .

In Figure 6, the disk with boundary L_4 is denoted by Y' satisfying $\text{Int } Y \subset Y'$ and $T(Y') \cap X = \emptyset$. Here $T = T_2 \circ T_1$. We define a smooth function $h : S^2 \rightarrow \mathbb{R}$ such that:

$$(5.1) \quad h(x) \begin{cases} = 0, & \text{if } x \in T(Y) \\ \in [0, 1], & \text{if } x \in T(Y') - T(Y) \\ = 1, & \text{if } x \in \overline{S^2 - T(Y')} \end{cases}$$

We also define a map $F : S^2 \rightarrow S^2$ such that:

$$(5.2) \quad F(x) = \begin{cases} g(x), & \text{if } x \in X \\ \phi_{h(x)}(x), & \text{if } x \in S^2 - X \end{cases}$$

By (5.1) and the properties of g , it is easy to show that F is a diffeomorphism on S^2 . Define a diffeomorphism H on S^2 by $H = T \circ F$.

By identifying $S^1 \times S^2$ with the quotient space $[0, 1] \times S^2 / (0, x) \sim (1, H(x))$, we define the flow ψ'_t by $\psi'_t(t_0, x) = (t_0 + t, x)$. It is easy to verify that ψ'_t is an NS flow on $S^1 \times S^2$ with e^+ closed orbit sources r^1, \dots, r^{e^+} , e^- closed orbit sinks a^1, \dots, a^{e^-} and a saddle basic set Λ with matrix A . Suppose $N(r^i)$ ($N(a^j)$) is a small standard neighborhood of r^i (a^j). Denote $S^1 \times S^2 - \cup_{i=1}^{e^+} N(r^i) - \cup_{j=1}^{e^-} N(a^j)$ by N and $\psi'_t|_N$ by ψ_t . By the properties of ψ'_t , it is easy to show that ψ_t satisfies (1), (2) and (3) of Lemma 5.4. \square

Theorem 5.5. *Let L be an abstract Lyapunov graph, then L is associated with an NS flow ϕ_t on $S^1 \times S^2$ such that each regular level set is separable if and only if the following conditions hold:*

- (1) *The underlying graph L is a tree with exactly one edge attached to each closed orbit sink or closed orbit source vertex. The weight of each edge of L is 1.*
- (2) *If a vertex is labeled with an SSFT with matrix $A_{n \times n}$, then*

$$0 < e^+ \leq k + 1, 0 < e^- \leq k + 1 \text{ and} \\ k \leq e^+ + e^-.$$

- (3) *There is at most 1 vertex satisfying that $k = e^+ + e^-$.*

Proof. The necessity. Let ϕ_t be an NS flow on $S^1 \times S^2$ such that each regular level set is separable. Suppose L is a Lyapunov graph of ϕ_t . By (1) in Proposition 4.4, the underlying graph L is a tree and the weight of each edge of L is 1. (2) and (3) follow from (2) in Proposition 4.6.

The sufficiency. Let L be a Lyapunov graph which satisfies (1), (2) and (3).

If there doesn't exist a vertex in L with $k = e^+ + e^-$. Then by Theorem 2.3 and the fact that we can paste $S^1 \times S^2$ up as two solid tori, it is easy to show that there exists an NS flow ϕ_t on $S^1 \times S^2$ with Lyapunov graph L .

If there exists a vertex in L with $k = e^+ + e^-$. We cut L along all the entrance edges and exit edges of V and paste e^+ closed orbit sink vertices $V_1^+, \dots, V_{e^+}^+$ and e^- closed orbit source vertices $V_1^-, \dots, V_{e^-}^-$ to the cut components of L without V suitably (similar to the surgery in Figure 2). Therefore, we obtain k abstract Lyapunov graphs: $L_1^+, \dots, L_{e^+}^+$ and $L_1^-, \dots, L_{e^-}^-$. Here L_i^+ and L_j^- contain vertices V_i^+ and V_j^- respectively. It is easy to check that L_i^+ and L_j^- satisfy the sufficient condition of Theorem 2.3 for any $i \in \{1, \dots, e^+\}$ and $j \in \{1, \dots, e^-\}$. Therefore, there exist NS flows φ_t^i and ψ_t^j with Lyapunov graphs L_i^+ and L_j^- respectively. Let a_i and r_j be the closed orbits associated with V_i^+ and V_j^- respectively. By the constructions in [7], we can suppose that a_i and r_j are all trivial knots.

For e^+ , e^- and A , we choose a compact 3-manifold N and an NS flow ψ_t in Lemma 5.4. Cut a standard neighborhood $N(a_i)$ of a_i and a standard neighborhood $N(r_j)$ of r_j and paste $S^3 - N(a_i)$ and $S^3 - N(r_j)$ to an entrance set of ∂N and an exit set of ∂N respectively. Doing the surgeries above for any $i \in \{1, \dots, e^+\}$ and $j \in \{1, \dots, e^-\}$, we obtain a closed 3-manifold M . a_i and r_j are all trivial knots. Moreover, N can be embedded into $S^1 \times S^2$ and there exists an embedding map $i : N \rightarrow S^1 \times S^2$ such that $\overline{S^1 \times S^2 - i(N)} \cong (e^+ + e^-)(S^1 \times D^2)$. Therefore, if the surgeries are suitable, then:

- (1) $M \cong S^1 \times S^2$;

- (2) φ_t^i on $S^3 - N(a_i)$, ψ_t^j on $S^3 - N(r_j)$ and ψ_t on N form an NS flow ϕ_t on $S^1 \times S^2$ with Lyapunov graph L

□

6. SINGULAR VERTEX

Basic definitions and facts about 3-manifolds can be found in [8].

A *filtrating neighborhood* [2] of a dimension 1 basic set K of an NS flow on a 3-manifold is a neighborhood U such that:

- (1) K is the maximal invariant set (for ϕ_t) in U .
- (2) The intersection of any orbit of ϕ_t with U is connected (in other terms, any orbit getting out of U never comes back).

In [2], F. Béguin and C. Bonatti proved that for a given dimension 1 basic set K , the filtrating neighborhood of K is unique up to topological equivalence. For an NS flow ϕ_t on a 3-manifold M , let $g : M \rightarrow R$ be a Lyapunov function associated to ϕ_t and L be a Lyapunov graph associated with g . Suppose K is a basic set of ϕ_t , $g(K) = y_0$ and K corresponds to the vertex $v \in L$. Then it is easy to show that the connected component $U \subset g^{-1}(y_0 - \epsilon, y_0 + \epsilon)$ which contains K is the filtrating neighborhood of K . This observation indicates that the topological structures of all Lyapunov graphs of an NS flow on a 3-manifold are the same. We also call the filtrating neighborhood of K the filtrating neighborhood associated with v .

Definition 6.1. Let L be a Lyapunov graph associated with an NS flow on a closed orientable 3-manifold M . If a vertex v of L satisfies that the filtrating neighborhood associated with v is not homeomorphic to a link complement, then we call the vertex v a *singular vertex*. Otherwise, we call the vertex v a *nonsingular vertex*.

Remark 6.2. Oka [10] also introduced the concept singular vertex. But our definition differs from his.

By a similar proof as in Proposition 4.6 (1), it is easy to show the following proposition.

Proposition 6.3. Let L be a Lyapunov graph associated with an NS flow on a closed orientable 3-manifold M . If V is a nonsingular vertex, then

$$\begin{aligned} 0 < e^+ \leq k + 1, 0 < e^- \leq k + 1 \text{ and} \\ k + 1 \leq e^+ + e^-. \end{aligned}$$

Proposition 6.3 tells us that if a vertex is a nonsingular vertex, the dynamics of the filtrating neighborhood associated with the vertex is similar to the dynamics of a filtrating neighborhood of an NS flow on S^3 . Therefore, we should study the filtrating neighborhood associated with a singular vertex.

Example 6.4. (1) Obviously, any vertex of a Lyapunov graph of an NS flow on S^3 is a nonsingular vertex.

- (2) The vertex v corresponds to the filtrating neighborhood N and the flow ψ_t in Lemma 5.4 is a singular vertex. In fact, for such a vertex v , by Lemma 5.4, $e^+ + e^- = k$. Therefore, v doesn't satisfy the necessary condition of a nonsingular vertex in Proposition 6.3.

- (3) For NS flows on $S^1 \times S^2$, by Theorem 5.1, Theorem 5.2 and Theorem 5.5, we have:

- (a) Let L be a Lyapunov graph associated with an NS flow on $S^1 \times S^2$. If the weight of any edge of L is 1, then there exists at most 1 singular vertex in L ;
- (b) For any $n \in \mathbb{N}$, there exists an NS flow ϕ_t such that there exist n singular vertices in L . Here L is a Lyapunov graph associated with ϕ_t .

Now we consider the number of the singular vertices in a Lyapunov graph associated with an NS flow on an irreducible, closed orientable 3-manifold.

By Dehn's lemma and some combinatorial surgeries, it is easy to show the following lemma.

Lemma 6.5. *Let T be a compressible torus in an irreducible, closed orientable 3-manifold M . Then T bounds a knot complement in M .*

Proposition 6.6. *Let L be a Lyapunov graph associated with an NS flow ϕ_t on an irreducible, closed orientable 3-manifold M . Then there exist at most $H(M) + 1$ singular vertices in L where $H(M)$ is the Haken number of M .*

Proof. For each edge e of L , we choose a regular level set Σ_e associated with e . Set $\Sigma = \{\Sigma_e | e \in L\}$. By Theorem 2.6, the weight of each edge of L is 1. Therefore, each Σ_e is homeomorphic to T^2 .

Let $\mathcal{T} = \{\mathcal{T} | \mathcal{T} = \{T_1, T_2, \dots, T_m\} \subset \Sigma\}$ where each T_i is incompressible and T_i is not parallel to T_j for any $i, j \in \{1, \dots, m\}$ and $i \neq j$. Obviously, $m \leq H(M)$. For any $\mathcal{T} = \{T_1, T_2, \dots, T_m\} \in \mathcal{T}$, let $d(\mathcal{T})$ be the number of the connected components of $M - (T_1 \sqcup T_2 \sqcup \dots \sqcup T_m)$. Obviously, $d(\mathcal{T}) \leq m + 1 \leq H(M) + 1$. Therefore, there exists $\mathcal{T}_0 = \{T_1, T_2, \dots, T_{m_0}\} \in \mathcal{T}$ satisfying $d(\mathcal{T}_0) = \max_{\mathcal{T} \in \mathcal{T}} (d(\mathcal{T})) \leq H(M) + 1$. Let $M_1, \dots, M_{d(\mathcal{T}_0)}$ be the components of $M - (T_1 \sqcup T_2 \sqcup \dots \sqcup T_{m_0})$. For any $\Sigma_e \in \Sigma - \mathcal{T}_0$, there are exactly the following two cases.

- (1) If Σ_e is incompressible in M , then Σ_e is parallel to T_i for some $i \in \{1, \dots, m\}$. Therefore, Σ_e and T_i bound a compact 3-manifold N_i which is homeomorphic to $T^2 \times [0, 1]$.
- (2) If Σ_e is compressible in M , by Lemma 6.5, then Σ_e bounds a knot complement $N \subset M$.

Suppose $\Sigma_e \subset M_j$ for some $j \in \{1, \dots, d(\mathcal{T}_0)\}$. In either of the above two cases, Σ_e divides M_j into two parts M_j^1 and M_j^2 such that one of M_j^1 and M_j^2 is a link component. Therefore, by some trivial inductive arguments, we can show that in M_i for any $i \in \{1, \dots, d(\mathcal{T}_0)\}$, there exists at most 1 filtrating neighborhood associated with a singular vertex. Noticing that $M_{d(\mathcal{T}_0)} \leq H(M) + 1$, we have that there exist at most $H(M) + 1$ singular vertices in L . \square

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DEPARTMENT OF MATHEMATICS, TONGJI UNIVERSITY, SHANGHAI, CHINA 20092